## Exercise 2D

**1** $x = a\cos\theta, y = b\sin\theta$

$\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$

So tangent is

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

Equation of tangent is $bx\cos\theta + ay\sin\theta = ab$

Normal gradient is $\frac{a\sin\theta}{b\cos\theta}$

So normal is $y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

Equation of normal is:

$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

**a** $a = 2, b = 1$

So equation of tangent is:

$x\cos\theta + 2y\sin\theta = 2$

Equation of normal is:

$2x\sin\theta - y\cos\theta = 3\sin\theta\cos\theta$

**b** $\frac{x^2}{25} + \frac{y^2}{9} = 1 \Rightarrow a = 5, b = 3$

So equation of tangent is:

$3x\cos\theta + 5y\sin\theta = 15$

Equation of normal is:

$5x\sin\theta - 3y\cos\theta = 16\sin\theta\cos\theta$

**2 a** $\frac{x^2}{9} + y^2 = 1 \Rightarrow \frac{2x}{9} + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x}{9y}$ so at $\left(\sqrt{5}, \frac{2}{3}\right)$ $m = -\frac{\sqrt{5}}{6}$

Tangent at

$\left(\sqrt{5}, \frac{2}{3}\right)$ is $y - \frac{2}{3} = -\frac{\sqrt{5}}{6}(x - \sqrt{5})$

$\Rightarrow 6y + \sqrt{5}x = 9$

Normal at

$\left(\sqrt{5}, \frac{2}{3}\right)$ is $y - \frac{2}{3} = \frac{6}{\sqrt{5}}(x - \sqrt{5})$

$3\sqrt{5}y - 2\sqrt{5} = 18x - 18\sqrt{5}$

$\Rightarrow 3\sqrt{5}y = 18x - 16\sqrt{5}$

**2 b** $\frac{x^2}{16} + \frac{y^2}{4} = 1 \Rightarrow \frac{x}{8} + \frac{y}{2}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x}{4y}$ so at $\left(-2, \sqrt{3}\right)$ $m = \frac{1}{2\sqrt{3}}$

Tangent at

$\left(-2, \sqrt{3}\right)$ is $y - \sqrt{3} = \frac{1}{2\sqrt{3}}(x - (-2))$

$\Rightarrow 2\sqrt{3}y - x = 8$

Normal at

$\left(-2, \sqrt{3}\right)$ is $y - \sqrt{3} = -2\sqrt{3}(x - (-2))$

$\Rightarrow y + 2\sqrt{3}x = -3\sqrt{3}$

**3** Use the chain rule to find the gradient:

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\cos t}{-a\sin t}$

Then the equation of the tangent is given by

$(y - b\sin t) = -\frac{b\cos t}{a\sin t}(x - a\cos t)$

$ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$

$bx\cos t + ay\sin t = ab$

**4 a** Using the method in Example 13, substitute for $y$ in the equation of the ellipse.

$\frac{x^2}{4} + y^2 = 1 \Rightarrow \frac{x^2}{4} + \left(x + \sqrt{5}\right)^2 = 1$

So $x^2 + 4\left(x^2 + 2\sqrt{5}x + 5\right) = 4$

$5x^2 + 8\sqrt{5}x + 16 = 0$

This has discriminant:

$\left(8\sqrt{5}\right)^2 - 4 \times 5 \times 16 = 0$

So the line meets the ellipse at only one point and therefore is a tangent to the ellipse.

**4 b** To find the point of contact, solve the equation from part **a** for $x$:

$5x^2+8\sqrt{5}x+16=0$

$\left(\sqrt{5}x+4\right)^2=0$

$\Rightarrow x=-\frac{4}{\sqrt{5}}=-\frac{4}{5}\sqrt{5}$

$\Rightarrow y=-\frac{4}{5}\sqrt{5}+\sqrt{5}=\frac{1}{5}\sqrt{5}$

So the point of contact is $\left(-\frac{4}{5}\sqrt{5},\frac{1}{5}\sqrt{5}\right)$

**5 a** $x=3\cos\theta, y=2\sin\theta \Rightarrow \frac{dy}{dx}=\frac{2\cos\theta}{-3\sin\theta}$

So gradient of normal is $\frac{3\sin\theta}{2\cos\theta}$

Equation of normal is:

$y-2\sin\theta=\frac{3\sin\theta}{2\cos\theta}(x-3\cos\theta)$

$2y\cos\theta-4\cos\theta\sin\theta$
$=3x\sin\theta-9\sin\theta\cos\theta$
$2y\cos\theta-3x\sin\theta$
$=-5\sin\theta\cos\theta$

**5 b** Normal at $P$ crosses the $x$-axis at

$y=0, x=-\frac{5}{6}$

Substituting into the equation for the normal from part **a**:

$\frac{15}{6}\sin\theta=-5\sin\theta\cos\theta$

$\Rightarrow \sin\theta\left(\frac{1}{2}+\cos\theta\right)=0$

$\Rightarrow \sin\theta=0 \text{ or } \cos\theta=-\frac{1}{2}$

$\sin\theta=0$ gives $\theta=0°$ or $180°$

$\cos\theta=-\frac{1}{2}$ gives $\theta=120°$ or $240°$

$\theta=0° \Rightarrow x=3, y=0$

$\theta=180° \Rightarrow x=-3, y=0$

$\theta=120° \Rightarrow x=\frac{3}{2}, y=\sqrt{3}$

$\theta=240° \Rightarrow x=-\frac{3}{2}, y=-\sqrt{3}$

So the coordinates of other possible positions of $P$ are

$(3,0),(-3,0)\left(-\frac{3}{2},\sqrt{3}\right)$ or $\left(-\frac{3}{2},-\sqrt{3}\right)$

**6** $y=mx+c$ is a tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

if $a^2m^2+b^2=c^2$

$y=2x+c \Rightarrow m=2, c=?$

$x^2+\frac{y^2}{4}=1 \Rightarrow a=1, b=2$

$a^2m^2+b^2=c^2 \Rightarrow 1\times4+4=c^2$

$c^2=8$ so $c=\pm2\sqrt{2}$

**7** Substitute $y = mx + 3$ into the equation for the ellipse.

$$x^2 + \frac{(mx+3)^2}{5} = 1$$

$$5x^2 + (mx+3)^2 = 5$$

$$(5+m^2)x^2 + 6mx + 4 = 0$$

Since the line is a tangent the discriminant of this equation must equal zero (must have equal roots).

So $36m^2 = 4\times(5+m^2)\times 4 = 80 + 16m^2$

$$20m^2 = 80$$

$$m^2 = 4$$

$$\therefore \quad m = \pm 2$$

> The $a^2m^2 + b^2 = c^2$ condition could be used as in question 6.

**8 a** $y = mx + 4,\ \frac{x^2}{3} + \frac{y^2}{4} = 1$

$\Rightarrow c = 4,\ a^2 = 3,\ b^2 = 4$

Using the condition $a^2m^2 + b^2 = c^2$:

$$a^2m^2 + b^2 = c^2$$

$$\Rightarrow 3m^2 + 4 = 16$$

$$3m^2 = 12$$

$$m = \pm 2$$

But $m > 0$, so $m = 2$

**b** $y = 2x + 4,\ \frac{x^2}{3} + \frac{y^2}{4} = 1$

Substitute into the equation for the ellipse:

$$\frac{x^2}{3} + \frac{(4x^2+16x+16)}{4} = 1$$

$$\Rightarrow x^2 + 3x^2 + 12x + 12 = 3$$

$$4x^2 + 12x + 9 = 0$$

$$(2x+3)^2 = 0$$

$x = -\frac{3}{2},\ y = 2x + 4 = 1$

So $P$ is $(-\frac{3}{2}, 1)$

**8 c** Gradient of normal is $-\frac{1}{2}$

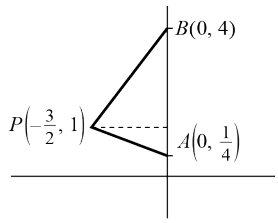
Equation of normal at $P$ is

$$y - 1 = -\frac{1}{2}\left(x - \left(-\frac{3}{2}\right)\right)$$

$$x = 0 \Rightarrow y = 1 - \frac{3}{4} = \frac{1}{4}$$

So $A$ is $(0, \frac{1}{4})$

**d** Tangent is $y = 2x + 4,\ x = 0 \Rightarrow y = 4$




So $B$ is $(0, 4)$

Area of $\Delta APB = \frac{1}{2}\left(4 - \frac{1}{4}\right)\times\frac{3}{2}$

$$= \frac{1}{2}\times\frac{15}{4}\times\frac{3}{2} = \frac{45}{16}$$

**9 a** $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\cos\theta,\ \frac{\mathrm{d}x}{\mathrm{d}\theta} = -3\sin\theta$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos\theta}{-3\sin\theta} = -\frac{2}{3}\cot\theta$$

**b** $\frac{\left(\frac{9}{5}\right)^2}{9} + \frac{\left(\frac{-8}{5}\right)^2}{4} = \frac{9}{25} + \frac{16}{25} = 1 = \text{RHS}$

So $Q\left(\frac{9}{5}, -\frac{8}{5}\right)$ lies on $E$

**c** Let $Q$ be the point $(3\cos\phi, 2\sin\phi)$

Using the coordinates of $Q$:

$$\frac{9}{5} = 3\cos\phi \Rightarrow \cos\phi = \frac{3}{5}$$

$$-\frac{8}{5} = 2\sin\phi \Rightarrow \sin\phi = -\frac{4}{5}$$

So $\cot\phi = -\frac{3}{4}$

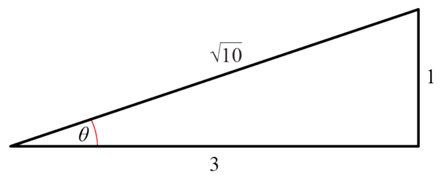
Gradient of tangent at $Q$ is

$$-\frac{2}{3}\cot\phi = -\frac{2}{3}\times-\frac{3}{4} = \frac{1}{2}$$

**9 d** Tangent at $P$ is perpendicular to tangent at $Q$, so gradient of tangent at $P = -2$

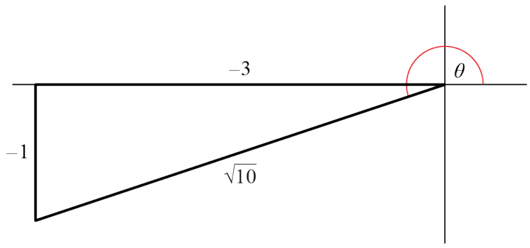
$$-2 = -\frac{2}{3}\cot\theta \Rightarrow \tan\theta = \frac{1}{3}$$

So $P$ is $\left(\frac{9}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$




or

$P$ is $\left(-\frac{9}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right)$




**10** $y = mx + c$ and $\frac{x^2}{9} + \frac{y^2}{46} = 1$

$\Rightarrow a^2 = 9,\ b^2 = 46$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 46 + 9m^2 = c^2$ **(1)**

$y = mx + c$ and $\frac{x^2}{25} + \frac{y^2}{14} = 1 \Rightarrow a^2 = 25,\ b^2 = 14$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 14 + 25m^2 = c^2$ **(2)**

**(1)** $-$ **(2)** $\Rightarrow 32 - 16m^2 = 0$

$\Rightarrow m^2 = 2$

$\therefore m = \pm\sqrt{2}$

$m^2 = 2$ and $14 + 25m^2 = c^2 \Rightarrow c^2 = 64$

$\therefore c = \pm 8$

So $m = \pm\sqrt{2},\ c = \pm 8$

**11** Use the chain rule to find the gradient of the tangent:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\cos\theta}{-8\sin\theta} = -\frac{\cos\theta}{2\sin\theta}$$

The equation of the tangent $l_1$ is given by

$$y - 4\sin\theta = -\frac{\cos\theta}{2\sin\theta}(x - 8\cos\theta)$$

$$2y\sin\theta - 8\sin^2\theta = -x\cos\theta + 8\cos^2\theta$$

Equation of the tangent is
$x\cos\theta + 2y\sin\theta = 8$

Gradient of the normal $l_2$ is $\frac{2\sin\theta}{\cos\theta}$ and its equation is

$$y - 4\sin\theta = \frac{2\sin\theta}{\cos\theta}(x - 8\cos\theta)$$

$$y\cos\theta - 4\sin\theta\cos\theta = 2x\sin\theta - 16\sin\theta\cos\theta$$

Equation of the normal is:
$2x\sin\theta - y\cos\theta = 12\sin\theta\cos\theta$

Line $l_1$ cuts the $x$-axis at $A$, so $y = 0$:
$x\cos\theta = 8$ so $x = 8\sec\theta$
$A$ is the point $(8\sec\theta, 0)$

Line $l_2$ cuts the $y$-axis at $B$, so $x = 0$:
$-y\cos\theta = 12\sin\theta\cos\theta$ so $y = -12\sin\theta$
$B$ is the point $(0, -12\sin\theta)$

Now find the equation of the line $AB$.

$$\frac{y - 0}{x - 8\sec\theta} = \frac{0 - (-12\sin\theta)}{8\sec\theta - 0}$$

$$\frac{y}{12\sin\theta} = \frac{x - 8\sec\theta}{8\sec\theta}$$

$$2y\sec\theta = 3x\sin\theta - 24\sec\theta\sin\theta$$

$$3x\sin\theta - 2y\sec\theta = 24\sec\theta\sin\theta$$

$$3x\sin\theta\cos\theta - 2y = 24\sin\theta$$

**12 a** Use the chain rule to find the gradient:

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3\cos\theta}{-5\sin\theta}$$

Then the equation of $l_1$ is given by

$$y-3\sin\theta=-\frac{3\cos\theta}{5\sin\theta}(x-5\cos\theta)$$

$$5y\sin\theta-15\sin^2\theta=-3x\cos\theta+15\cos^2\theta$$

$$3x\cos\theta+5y\sin\theta=15$$

**b** At $Q$ the line cuts the $y$-axis, so $x = 0$
Substitute in the equation for line $l_1$:

$$5y\sin\theta=15 \text{ so } y=\frac{3}{\sin\theta}$$

The point $Q$ has coordinates $\left(0,\frac{3}{\sin\theta}\right)$

The gradient of any line perpendicular to $l_1$ is:

$$\frac{5\sin\theta}{3\cos\theta}$$

Then the equation of $l_2$ is:

$$y-\frac{3}{\sin\theta}=\frac{5\sin\theta}{3\cos\theta}x$$

$$3y\sin\theta\cos\theta-9\cos\theta=5x\sin^2\theta$$

$$5x\sin^2\theta-3y\sin\theta\cos\theta=-9\cos\theta$$

**c** If $l_2$ cuts the $x$-axis at $(-4, 0)$, then substituting into the equation for $l_2$ gives

$$-20\sin^2\theta=-9\cos\theta$$

$$20(1-\cos^2\theta)=9\cos\theta$$

$$20-20\cos^2\theta=9\cos\theta$$

$$20\cos^2\theta+9\cos\theta-20=0$$

Using the quadratic formula to solve gives:

$$\cos\theta=\frac{-9\pm\sqrt{81-4\times20\times(-20)}}{40}=\frac{-9\pm41}{40}$$

Obviously $\cos\theta\neq-\frac{52}{40}<-1$, so

$$\cos\theta=\frac{32}{40}=\frac{4}{5}$$

**13 a** Use the chain rule to find the gradient:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4\cos t}{-2\sin t}=-\frac{2\cos t}{\sin t}$$

Then an equation for $l_1$ is given by:

$$y-4\sin t=-\frac{2\cos t}{\sin t}(x-2\cos t)$$

$$y\sin t-4\sin^2 t=-2x\cos t+4\cos^2 t$$

$$2x\cos t+y\sin t=4$$

**b** Since $l_2$ is perpendicular to $l_1$, the gradient of $l_2$ is $\frac{\sin t}{2\cos t}$

The line passes through the origin, so the equation is $y=mx\Rightarrow y=x\frac{\sin t}{2\cos t}$

Substituting $y=x\frac{\sin t}{2\cos t}$ into the equation of $l_1$ to find the coordinates of the intersection gives:

$$2x\cos t+x\frac{\sin^2 t}{2\cos t}=4$$

$$\Rightarrow 4x\cos^2 t+x\sin^2 t=8\cos t$$

$$\Rightarrow x=\frac{8\cos t}{4\cos^2 t+\sin^2 t}$$

Then $y=\frac{\sin t}{2\cos t}\times\frac{8\cos t}{4\cos^2 t+\sin^2 t}=\frac{4\sin t}{4\cos^2 t+\sin^2 t}$

The coordinates of $Q$ are:

$$\left(\frac{8\cos t}{4\cos^2 t+\sin^2 t},\frac{4\sin t}{4\cos^2 t+\sin^2 t}\right)$$